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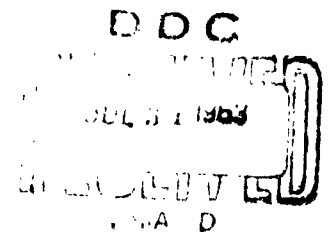
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EXISTENCE AND UNIQUENESS THEOREMS  
IN INVARIANT IMBEDDING-II:  
CONVERGENCE OF A NEW  
DIFFERENCE ALGORITHM

Richard Bellman and Kenneth L. Cooke

PREPARED FOR:  
ADVANCED RESEARCH PROJECTS AGENCY



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*The* **RAND** *Corporation*  
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## PREFACE

Part of the RAND research program consists of basic supporting studies in mathematics. This Memorandum is the second in a series dealing with a number of rigorous aspects of the highly useful mathematical theory known as invariant imbedding. In this theory invariance principles are applied to handle a variety of conceptual and computational aspects of mathematical physics.

The research presented here was sponsored by the Advanced Research Projects Agency.

### SUMMARY

The theory of invariant imbedding leads to new types of difference approximations to partial differential equations. To illustrate the type of analysis required to establish the convergence of the solution of one to the solution of the other, we consider the difference approximation  $u(x + \Delta, y) = u(x, y + u(x, y)\Delta)$ , and the limiting equation  $u_x = uu_y$ .

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## EXISTENCE AND UNIQUENESS THEOREMS IN INVARIANT IMBEDDING - II: CONVERGENCE OF A NEW DIFFERENCE ALGORITHM

### 1. INTRODUCTION

In the first paper of this series, we indicated how a mathematical model of a "transport process," constructed according to the theory of invariant imbedding, led to an interesting system of nonlinear ordinary differential equations of Riccati type, and we showed how a new type of conservation principle could be used to establish existence of solutions.

If interactions between particles are admitted, in addition to the already existing interaction between particles and the medium, then invariant imbedding techniques lead to partial differential equations of hyperbolic type. The stratification used in the derivation of these equations, following imbedding procedures, automatically produces a set of approximating difference equations of novel form, quite different from the usual equations of this type; see [1], where other references may be found.

Since the discussion of the equations arising from invariant imbedding is rather complex, we shall illustrate the method first with an analogous treatment of the equation

$$(1.1) \quad \begin{aligned} u_x + uu_y &= 0, & -\infty < y < \infty, & \quad x \geq 0, \\ u(0, y) &= g(y), \end{aligned}$$



an equation of some interest in itself. We shall consider an approximating difference equation

$$(1.2) \quad u(y, x + \Delta) = u(y - u(y, x)\Delta, x),$$

$x = 0, \Delta, 2\Delta, \dots, -\infty < y < \infty$ , an equation which we know yields excellent numerical results; [2], [3]. For  $x \neq n\Delta$ ,  $u(y, x)$  is defined by linear interpolation.

Equation (1.1) is an exceedingly useful nonlinear partial differential equation, since it possesses an explicit solution (in implicit form,  $u = g(x + uy)$ ) that exhibits a shock phenomenon. Hence, it is a very handy test of proposed numerical procedures.

Looking at Eqs. (1.1) and (1.2) there are several questions we can ask:

- (1.3) a. As  $\Delta \rightarrow 0$ , does the solution of (1.2) approach the solution of (1.1), assuming that we have already established the existence of a unique solution of (1.1) ?
- b. Can we use (1.2) to establish the existence of a solution of (1.1) ?
- c. Let  $\Delta \rightarrow 0$  through a sequence  $\{a/2^n\}$  and call the function obtained from (1.2), with  $\Delta = a/2^n$ ,  $u_n(x, y)$ . Does the sequence  $\{u_n(x, y)\}$  converge, independently of the existence of solutions of (1.1) ?

These questions are of computational significance as well as analytic interest. We shall discuss (a) and (c) here, and reserve a discussion of (b) for a later paper devoted to the more complex equations associated with transport processes.

In passing, let us refer the reader to [8] where the problem of obtaining higher order approximations of the general form of (1.2) is briefly discussed.

## 2. PRELIMINARIES

Let us introduce the following notation:

$$(2.1) \quad u_n(y) = u(n\Delta, y), \quad u'_n(y) = \frac{\partial}{\partial y} u(n\Delta, y),$$

and the norms

$$(2.2) \quad \begin{aligned} u_n &= \|u_n(y)\| = \max_y |u_n(y)|, \\ u'_n &= \|u'_n(y)\| = \max_y |u'_n(y)|, \\ c_1 &= \max_y |g(y)|. \end{aligned}$$

We assume that  $g(y)$  is continuous for all  $y$ , and that  $c_1 < \infty$ .

The recurrence relation of (1.2) assumes the form

$$(2.3) \quad u_{n+1}(y) = u_n(y - u_n(y)\Delta), \quad n = 0, 1, 2, \dots,$$

$u_0(y) = g(y)$ . This is a difference equation which is more closely related to the functional equation studied by Myskis [4] than are the

conventional differential equations whose study was initiated by Courant, Friedrichs and Lewy; see [5] for further references.

Differentiating with respect to  $y$ , we have

$$(2.4) \quad u'_{n+1}(y) = u'_n(y - \Delta u_n(y)) (1 - \Delta u'_n(y)).$$

Upon taking absolute values we are led to the following inequality between norms:

$$(2.5) \quad u'_{n+1} \leq u'_n (1 + \Delta u'_n),$$

a Riccati difference inequality.

For our further purposes, we require the following simple result.

Lemma. Let  $\{u_n\}$  be a sequence of nonnegative numbers satisfying the relation

$$(2.6) \quad u_{n+1} \leq c\Delta + u_n(1 + 2c\Delta + c\Delta u_n), \quad n = 0, 1, \dots$$

Then if  $\Delta$  is sufficiently small, viz.,

$$(2.7) \quad cn\Delta < \frac{1}{1 + u_0},$$

it follows that

$$(2.8) \quad u_n \leq \left[ \frac{1}{1 + u_0} - cn\Delta \right]^{-1} - 1.$$

Proof. Let  $\{v_n\}$  be defined by the equation

$$(2.9) \quad v_{n+1} = c\Delta + v_n(1 + 2c\Delta + c\Delta v_n), \quad v_0 = u_0.$$

It is clear that  $v_{n+1} \geq v_n$  and  $v_n \geq u_n$  for  $n = 0, 1, 2, \dots$ .

Then

$$(2.10) \quad \frac{v_{n+1} - v_n}{(1 + v_n)(1 + v_{n+1})} = c\Delta \frac{(1 + v_n)}{(1 + v_{n+1})} \leq c\Delta.$$

Hence

$$(2.11) \quad \sum_{i=0}^{n-1} \frac{v_{i+1} - v_i}{(1 + v_i)(1 + v_{i+1})} = \frac{1}{1 + v_0} - \frac{1}{1 + v_n} \leq cn\Delta,$$

a result which implies (2.8).

Returning to (2.5), we see that  $\{u_n\}$  satisfies (2.6), with

$c = 1$ , whence

$$(2.12) \quad u'_n \leq \left[ \frac{1}{1 + u_0} - n\Delta \right]^{-1} - 1,$$

provided that  $n\Delta \leq 1/(1 + u_0)$ .

Keeping the  $x$ -interval small enough,  $n\Delta \leq a \leq 1$ , we obtain a bound on  $u'_n$  which is independent of  $n$  and  $\Delta$ .

### 3. PROOF OF CONVERGENCE TO KNOWN SOLUTION

Let us now turn to the discussion of the convergence of the solution of (1.2) under the hypothesis that we have already established the existence of a solution in an initial  $x$ -interval  $0 \leq x \leq a$ , a matter

easily accomplished in this case and indeed explicitly by means of the implicit relation mentioned in Sec. 1.

Let  $v(x, t)$  represent this solution which possesses continuous derivatives in the region  $0 \leq x \leq a$ ,  $-\infty < y < \infty$ , if  $a$  is small. Then we have the equation

$$(3.1) \quad v(x + \Delta, y) = v(x, y - \Delta v(x, y)) + o(\Delta),$$

uniformly in this region. Set

$$(3.2) \quad u_n(y) = u(n\Delta, y), \quad v_n(y) = v(n\Delta, y).$$

Then (3.1) and (2.3) yield the relation

$$\begin{aligned} (3.3) \quad u_{n+1}(y) - v_{n+1}(y) &= u_n(y - \Delta u_n(y)) - v_n(y - \Delta v_n(y)) + o(\Delta) \\ &= u_n(y - \Delta v_n(y)) - v_n(y - \Delta v_n(y)) \\ &\quad - u_n(y - \Delta v_n(y)) + u_n(y - \Delta u_n(y)) + o(\Delta). \end{aligned}$$

Introduce the sequence of constants  $\{r_n\}$ , where

$$(3.4) \quad r_n = \max_y |u_n(y) - v_n(y)|.$$

Then (3.3) yields

$$\begin{aligned} |u_{n+1}(y) - v_{n+1}(y)| &\leq |u_n(y - \Delta v_n(y)) - v_n(y - \Delta v_n(y))| \\ (3.5) \quad &+ |u_n(y - \Delta v_n(y)) - u_n(y - \Delta u_n(y))| \\ &+ o(\Delta), \end{aligned}$$

or

$$(3.6) \quad \begin{aligned} r_{n+1} &\leq r_n + \Delta r_n u'_n + \epsilon \Delta, \\ &\leq r_n (1 + c_1 \Delta) + \epsilon \Delta \end{aligned}$$

(where  $\epsilon \rightarrow 0$  as  $\Delta \rightarrow 0$ ), because of the uniform bound on  $u'_n$  established in Sec. 2 above. It is easy to show inductively that this implies

$$(3.7) \quad r_n \leq \frac{\epsilon}{c_1} (e^{c_1 n \Delta} - 1) \leq \frac{\epsilon}{c_1} (e^{c_1 a} - 1),$$

since  $n \Delta \leq a$ . It follows that  $r_n \rightarrow 0$  as  $\Delta \rightarrow 0$ , and thus that

$$(3.8) \quad \lim_{\Delta \rightarrow 0} (u(x, y) - v(x, y)) = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq a.$$

Hence, if a solution of the presented type exists, the sequence defined by (2.3) converges to it. Furthermore, the foregoing establishes the uniqueness of such a solution.

#### 4. PROOF OF CONVERGENCE OF DISCRETE SEQUENCE

Let us now turn to a demonstration of the convergence of the sequence  $\{u_n(y)\}$  in the case where we do not presuppose the existence of a solution of the partial differential equation. Let  $a > 0$ , and choose successively  $\Delta = a, a/2, a/4, \dots$ . Let  $u(x, y)$  be an approximating function determined by a particular  $\Delta$  in this sequence and let  $\hat{u}(x, y)$  be the corresponding function for  $\Delta/2$ .

As before, we let

$$(4.1) \quad u_n(y) = u(n\Delta, y), \quad \hat{u}(y) = \hat{u}(n\Delta, y).$$

Now let us compare these two functions. We have

$$(4.2) \quad u_{n+1}(y) = u_n(y - \Delta u_n(y))$$

and, using (2.3) twice,

$$(4.3) \quad \begin{aligned} \hat{u}_{n+1}(y) &= \hat{u}\left[\left(n + \frac{1}{2}\right)\Delta, y - \frac{\Delta}{2} \hat{u}\left(\left(n + \frac{1}{2}\right)\Delta, y\right)\right] \\ &= \hat{u}_n\left(y - \frac{\Delta}{2} \hat{u}_n(y')\right), \end{aligned}$$

where

$$(4.4) \quad y' = y - \frac{\Delta}{2} \hat{u}\left(\left(n + \frac{1}{2}\right)\Delta, y\right).$$

Since  $y' = y + O(\Delta)$ , we have

$$(4.5) \quad \hat{u}_n(y') = \hat{u}_n(y) + O(\Delta),$$

using the uniform boundedness of  $\hat{u}$  and  $\hat{u}_y$ , as shown above.

Furthermore,

$$(4.6) \quad \hat{u}\left(\left(n + \frac{1}{2}\right)\Delta, y\right) = \hat{u}_n\left(y - \frac{\Delta}{2} \hat{u}_n(y)\right) = \hat{u}_n(y) + O(\Delta).$$

Hence

$$(4.7) \quad y' = y - \frac{\Delta}{2} \hat{u}_n(y) + O(\Delta^2),$$

where the 0-term is uniform in  $x$ ,  $y$ , and  $\Delta$ .

Equations (4.7) and (4.2) enable us to compare the functions  $u$  and  $\hat{u}$  at  $x = 0, \Delta, 2\Delta, \dots$ . We shall prove that there exist positive numbers  $a$  and  $k$  such that

$$(4.8) \quad \max_y |u_n(y) - \hat{u}_n(y)| \leq kn\Delta^2,$$

for  $n\Delta = 0, \Delta, \dots, a$ . Proceeding inductively, suppose that it holds for  $n$ . Using (4.2) and (4.7), we have

$$\begin{aligned} (4.9) \quad |u_{n+1}(y) - \hat{u}_{n+1}(y)| &\leq |u_n(y - \Delta u_n(y)) - \hat{u}_n(y - \Delta u_n(y))| \\ &\quad + |\hat{u}_n(y - \Delta u_n(y)) - \hat{u}_n(y - \Delta \hat{u}_n(y))| \\ &\quad + c_1 \Delta^2 \\ &\leq kn\Delta^2 + c_1 \Delta^2 \\ &\quad + |\hat{u}_n(y - \Delta u_n(y)) - \hat{u}_n(y - \Delta \hat{u}_n(y))| \\ &\leq kn\Delta^2 + c_1 \Delta^2 + c_2 \Delta |u_n(y) - \hat{u}_n(y)| \\ &\leq kn\Delta^2 + c_1 \Delta^2 + c_2 \Delta (kn\Delta^2). \end{aligned}$$

Since  $n\Delta \leq a$ , we can write (4.9) as

$$(4.10) \quad |u_{n+1}(y) - \hat{u}_{n+1}(y)| \leq kn\Delta^2 + c_1 \Delta^2 + c_2 \Delta (ka\Delta).$$

Hence if  $k$  and  $a$  are chosen so that



$$(4.11) \quad c_1 + c_2 ka < k,$$

which means  $c_2 a < 1$ ,  $k > c_1$ , the induction is complete.

Since  $u(x, y)$  and  $\hat{u}(x, y)$  have been defined by linear interpolation for  $x$  not a lattice point, we have

$$(4.12) \quad \max_y |u(x, y) - \hat{u}(x, y)| \leq c_3 \Delta, \quad 0 \leq x \leq a.$$

If we let  $u_r(x, y)$  denote the approximating function corresponding to  $\Delta = a/2^r$ ,  $r = 1, 2, \dots$ , we obtain from (4.12) the estimate

$$(4.13) \quad \sum_{r=n}^{\infty} |u_{r+1}(x, y) - u_r(x, y)| \leq \sum_{r=n}^{\infty} \frac{c_3 a}{2^{r+1}} < \frac{c_3 a}{2^n},$$

from which we deduce that

$$(4.14) \quad \lim_{n \rightarrow \infty} u_n(x, y) = u(x, y)$$

exists uniformly for  $0 \leq x \leq a$ ,  $-\infty < y < \infty$ .

## 5. GENERALIZED SOLUTIONS

In the case of the general equations of invariant imbedding [6], we are interested in establishing the limiting behavior of equations obtained from an infinitesimal stratification of a medium. The procedure we have used above enables us to define a "generalized solution" in a new sense, an idea already put forth in [7], in connection with the application of dynamic programming to the calculus of variations.

In subsequent papers we shall discuss the utility of these new difference schemes in connection with establishing the existence of a solution of the corresponding hyperbolic partial differential equations of invariant imbedding from first principles.

## REFERENCES

1. Noh, W. F., and M. H. Protter, "Difference Methods and the Equations of Hydrodynamics," J. Math. and Mech., Vol. 12, 1963, pp. 149-192.
2. Bellman, R. E., I. Cherry, and G. M. Wing, "A Note on the Numerical Integration of a Class of Nonlinear Hyperbolic Equations," Q. Appl. Math., Vol. 16, 1958, pp. 181-183.
3. Bellman, R. E., R. Kalaba, and B. Kotkin, "On a New Approach to the Computational Solution of Partial Differential Equations," Proc. Nat. Acad. Sci. USA, Vol. 48, 1962, pp. 1325-1327.
4. Myskis, A. D., "Linear Differential Equations with a Delayed Argument," Gos Izd. Tekhniko-Teoret. Lit., Moscow, 1951.
5. Bellman, R. E. and K. L. Cooke, Differential-Difference Equations, Academic Press, New York, 1963.
6. Bellman, R. E., R. Kalaba and G. M. Wing, "Invariant Imbedding and Mathematical Physics - I: Particle Processes," J. Math. Phys., Vol. 1, 1960, pp. 280-308.
7. Bellman, R. E., "Functional Equations in the Theory of Dynamic Programming - VI: A Direct Convergence Proof," Ann. Math., Vol. 65, 1957, pp. 215-223.
8. Bellman, R., "Some Questions Concerning Difference Approximations to Partial Differential Equations," Boll. d'Unione Matematico, Vol. 17, 1962, pp. 188-190.